# The four-fermion interaction in D=2,3,4: a nonperturbative treatment.

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**Abstract**

A new nonperturbative approach is used to investigate the Gross-Neveu model of four fermion interaction in the space-time dimensions 2, 3 and 4, the number $N$ of inner degrees of freedom being a fixed integer. The spontaneous symmetry breaking is shown to exist in $D = 2, 3$ and the running coupling constant is calculated. The four dimensional theory seems to be trivial.

## 1 Introduction

In spite of its great and numerous successes the perturbation theory cannot describe a wide class of important phenomena (like confinement or spontaneous symmetry breaking) playing the key role in the problem of full and comprehensive description of physical reality. So it is quite natural that considerable efforts are applied in order to develop the nonperturbative methods in the quantum field theory. So far, only a few sufficiently effective methods like the effective potential, the $1/N$ expansion, the Gauss effective potential method [1] or the variational perturbation theory [2] are known.

A new nonperturbative approach has been recently proposed in work [3]. The ability of the method was demonstrated by the example of a self interacting scalar field in various dimensions. In the present paper we would like to investigate the Gross-Neveu model of the fermion fields with an arbitrary *fixed* number $N$ of inner degrees of freedom. The cases $D = 2, 3, 4$ were elaborated and the spontaneous symmetry breaking was found to exist in two and three dimensions. For the four dimensional Gross-Neveu model our consideration gives the arguments in favor of the triviality of this model. These results exhibit the efficiency of the method and they are the finite $N$ generalization of the known results obtained in the framework of $1/N$ expansion [4]–[8].

The paper is organized as follows. Section 2 consists of a short introduction to the method of [3] to be used throughout the paper. Sections 3, 4 and 5 are devoted to the Gross-Neveu model in the two, three and four dimensions, respectively. Section 6 contains another approach to the subject based on the bilocal source formalism.

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## 2 The method

In our approach we will use one of the most suitable tools for the nonperturbative treatment of a quantum field model — the Schwinger-Dyson equation. As the system of Schwinger-Dyson equations for the Green functions consists of the infinite number of mutually connected equations, one should truncate it in some way in order to find an approximate solution. It is obvious enough that the concrete way of truncation has a crucial significance for the results. For example, if we solve the Schwinger-Dyson system iteratively by expanding the Green functions into the series in the coupling constant, we obtain the perturbative solution. Being the simplest from the practical point of view the perturbative approach is the worst in the mathematical sense. The matter is that a small parameter (the coupling constant) is a multiplier at the highest derivative term of the functional differential Schwinger-Dyson equation for the Green functions generating functional. This means that the equation is a singularly perturbed one. So, the perturbative procedure is valid only for the restrictive class of boundary conditions and cannot, in principle catch all solutions [9].

Taking into account the reasons above, we can conclude that a proper approximation scheme for the nonperturbative solution of the Schwinger-Dyson equation should obey the following requirements:

- It must take into account the highest derivative term of the Schwinger-Dyson equation already at the leading approximation.
- It must allow one to make the renormalization procedure.
- It should be simple enough for the practical calculations.

An approximation scheme proposed in [3] obeys the requirements listed above. Below we give a short introduction to the main ideas of the method for the reader's convenience.

Consider the theory of a self-interacting scalar field $\phi(x)$ with the action

$$S(\phi) = \int dx \left( \frac{1}{2}(\partial_\mu \phi)^2 - \frac{\mu^2}{2}\phi^2 - \lambda\, \phi^4 \right) . \tag{2.1}$$

The generating functional of $n$-point Green functions can be written as follows

$$G = \sum_{n=0}^{\infty} G_n j^n , \tag{2.2}$$

where $j(x)$ is a field source. The $n$-th derivative of G at $j = 0$ is the $n$-point Green function $G_n$.

The Schwinger-Dyson equation for the generating functional of this model reads

$$(\mu^2 + \partial^2)\frac{\delta G}{\delta j(x)} + 4\lambda \frac{\delta^3 G}{\delta j^3(x)} - ij(x)G = 0 \; . \tag{2.3}$$

The central idea of the iterative scheme is to consider the *last* term of this equation as a perturbation to the leading approximation. That is we take the following "equation with constant coefficients" as the leading approximation:

$$(\mu^2 + \partial^2)\frac{\delta G^{(0)}}{\delta j(x)} + 4\lambda \frac{\delta^3 G^{(0)}}{\delta j^3(x)} = 0 \; . \tag{2.4}$$

Presenting the full functional $G(j)$ as the sum

$$G(j) = \sum_n G^{(n)}(j) \tag{2.5}$$

we then write for the terms of this sum the recursive chain of equations:

$$(\mu^2 + \partial^2)\frac{\delta G^{(n)}}{\delta j(x)} + 4\lambda\frac{\delta^3 G^{(n)}}{\delta j^3(x)} = ij(x)G^{(n-1)} \ . \tag{2.6}$$

The solution for (2.4) is sought for in the form $G^{(0)} = \exp{(i\sigma j)}$ and from (2.4) one obtains a characteristic equation for the function $\sigma(x)$. For the $G^{(n)}$ we put $G^{(n)} = P_n(j)G^{(0)}$, where $P_n(j)$ is a polynomial in $j(x)$ with unknown coefficients to be defined from (2.6). These coefficients define the Green functions of the corresponding step of the iteration scheme. It should be noted, that at the leading approximation one defines "the vacuum" of the model, at the first step the connected part of the propagator enters the game, whereas for the higher Green functions one can define an approximant for the disconnected part only. At the next steps of the scheme we calculate many-particle amplitudes and corrections to the propagator and so on.

This procedure is model independent and has a regular character. The last property is due to the generating functional is regular at $j = 0$ *by definition*, as its derivatives at this point are the Green functions of the model. Therefore the perturbation theory around the point $j = 0$ is *regular* recipe for solving the Schwinger-Dyson equation.

The renormalization procedure can be easily introduced into the scheme (see [3] for details). Shortly the renormalization can be carried out in the following few steps.

**(i)** All the necessary counter terms are expanded in a sum, analogous to (2.5):

$$\mu^2 \rightarrow \mu^2 + \delta\mu^2_{(0)} + \delta\mu^2_{(1)} + \ \ldots \qquad \lambda \rightarrow \lambda + \delta\lambda_{(0)} + \delta\lambda_{(1)} + \ \ldots \quad \text{etc.,} \tag{2.7}$$

where the subscript of a counter term stands for the approximation scheme step at which this counter term should be taken into account.

**(ii)** Equations (2.4) and (2.6) are modified respectively:

$$(\mu^2 + \delta\mu^2_{(0)} + (1 + \delta Z^{(0)}_\phi)\,\partial^2)\frac{\delta G^{(0)}}{\delta j(x)} + 4(\lambda + \delta\lambda_{(0)})\frac{\delta^3 G^{(0)}}{\delta j^3(x)} = 0 \tag{2.8}$$

$$\begin{aligned}(\mu^2 + \delta\mu^2_{(0)} + (1 + \delta Z^{(0)}_\phi)\,\partial^2)\frac{\delta G^{(1)}}{\delta j(x)} \ &+ \ 4(\lambda + \delta\lambda_{(0)})\frac{\delta^3 G^{(1)}}{\delta j^3(x)} \\ &= \ ij(x)G^{(0)} - \delta\mu^2_{(1)}\frac{\delta G^{(0)}}{\delta j(x)} - \delta Z^{(1)}_\phi \partial^2 \frac{\delta G^{(0)}}{\delta j(x)}\end{aligned} \tag{2.9}$$

and so on.

**(iii)** The counter terms entering the scheme at the $n$-th step can be fixed only at the next step. Before the renormalization of the next step has been done, the equations of the $n$-th step are nothing but some relations among the counter terms.

Now we are ready to go to the subject of our paper — the Gross-Neveu model of the four fermion interaction [4].

# 3 The general consideration and D=2 case

In [4] D. Gross and A. Neveu have investigated the model of the $N$-coloured spinor fields with scalar-scalar four fermion interaction at large $N$. Most of the works devoted to the Gross-Neveu model also deals with the $1/N$ expansion [5]–[7]. Our goal is to investigate the Gross-Neveu model when the number of colours $N$ is an arbitrary fixed integer.

The model is defined in the $D$-dimensional Minkowski space-time by the action:

$$S = \int d^D x \left( \bar{\psi}\,(i\,\partial\!\!\!/ - m)\,\psi + \frac{\lambda}{2}\,(\bar{\psi}\psi)^2 \right), \tag{3.1}$$

where $\psi_k(x)$ is a spinor field with $N$ isotopic degrees of freedom. The summation over the isotopic indices $k$ is implicit in (3.1). The mass dimension of the coupling constant $\lambda$ is zero in the two dimensional space-time, therefore model (3.1) is renormalizable in the two dimensions.

Let us transform our model to that of the spinor and scalar fields coupled by Yukawa interaction [4]:

$$S_{\text{eff}} = \int d^D x \left( \bar{\psi}\,(i\,\partial\!\!\!/ - m)\,\psi - \frac{1}{2}\phi(\mu^2 + \partial^2)\phi + \beta\,\phi(\bar{\psi}\psi) \right) . \tag{3.2}$$

Model (3.2) is equivalent to (3.1) if one identifies $\beta = \mu\sqrt{\lambda}$, $\phi \to \phi/\mu$ and takes the limit $\mu \to \infty$. To be more exact, we define the renormalized Gross-Neveu model as $\mu \to \infty$ limit of the renormalized effective model (3.2), $\mu$ being the renormalized mass of $\phi$ (see below). Below we will refer to this identification as the Gross-Neveu limit. The simplest way to verify the mentioned equivalence is to consider the path integral representation for the generating functional:

$$G_{\text{eff}}(j) = \int \mathcal{D}\phi\mathcal{D}\bar{\psi}\mathcal{D}\psi\ \exp\left( iS_{\text{eff}} + i\int j\phi \right) . \tag{3.3}$$

Now we perform the Gauss integration over the spinor field in (3.3) and then apply the just described approximation scheme to the resulting functional $G(j)$. However there is one subtle point about the case $N = 1$ in two dimensions. As is known, the two-dimensional Gross-Neveu model at $N = 1$ is equivalent to the Thirring model. It is known, that the Thirring model does not reveal the spontaneous chiral symmetry breaking, so the same should be true for the Gross-Neveu model at $N = 1$. The matter is that in the case of two dimensions and $N = 1$ (and *only* in this case) the character of the symmetry of the Gross-Neveu model changes. At the chiral limit $m \to 0$ this model is invariant under the discrete transformation $\psi \to \gamma_5\psi$. But in the exclusive case $N = 1$ the symmetry becomes continuous: $\psi \to e^{i\alpha\gamma_5}\psi$. Such an effect can be easily seen by passing to the spinor components: we find that non-invariant terms are equal to zero as a consequence of the grassmanian nature of the spinor field. So we have the continuous symmetry in the two dimensions, which cannot be broken spontaneously due to the Mermin-Wagner-Coleman theorem.

In order to take into account this fact explicitly one could (following, for example, [5]) integrate over only $(N-1)$ components of the spinor field in the generating functional (3.3). The consideration will be slightly more complicated, but no new effect will arise except for the multiplier $(N-1)$ instead of $N$ in front of all the quantities responsible for the chiral symmetry breaking. We choose the more transparent way of integrating over all the components of the spinor field but should always keep in mind that our consideration is valid only for $N \geq 2$.

Upon integrating over the spinor field in (3.3) we find the following Schwinger-Dyson equation for the generating functional $G(j)$:

$$(\mu^2+\partial^2)\frac{\delta G}{\delta j(x)}+N\beta\int d^Dy\,\mathrm{Tr}\left[\Big(1-\beta\triangle\cdot\frac{\delta}{i\delta j}\Big)^{-1}(x,y)\triangle(y-x)\right]G-ij(x)G=0\,, \quad (3.4)$$

where $\triangle(x)$ is the free fermion propagator:

$$(i\,\partial\!\!\!/-m)\triangle(x)=-\,\delta(x)\quad,\quad\triangle(x)=\int\frac{d^Dp}{(2\pi)^D}\frac{p\!\!\!/+m}{m^2-p^2}\,,$$

and $D$ stands for the space-time dimension.

Writing the generating functional as the sum $G=G^{(0)}+G^{(1)}+\ldots$, we put for the leading approximation $G^{(0)}$:

$$(\mu^2+\partial^2)\frac{\delta G^{(0)}}{\delta j(x)}+N\beta\int d^Dy\,\mathrm{Tr}\left[\Big(1-\beta\triangle\cdot\frac{\delta}{i\delta j}\Big)^{-1}(x,y)\triangle(y-x)\right]G^{(0)}=0\,. \quad (3.5)$$

The solution to this equation is sought for in the form $G^{(0)}=\exp(i\sigma*j)$, where $*$ means the space-time integration and $\sigma$ must be constant by virtue of the Poincaré invariance. Therefore the Green functions of the leading approximation are constants and fully disconnected. Nontrivial connected parts of the Green functions arise at the further steps of iterations (see discussion in Section 2).

Now we have from (3.5):

$$i\mu^2\sigma+N\beta\,\mathrm{Tr}R(0)=0\,, \quad (3.6)$$

where we adopted the convenient notation $R(x)$:

$$R(x)\equiv\int d^Dy\big(1-\beta\sigma\triangle\big)^{-1}(x-y)\triangle(y)\equiv(\triangle^{-1}-\beta\sigma)^{-1}(x)\,.$$

The Fourier transformation of the function $R$ reads:

$$R(x)=\int\frac{d^Dp}{(2\pi)^D}e^{-ipx}\frac{p\!\!\!/+M}{M^2-p^2}\,,\qquad M\equiv m-\beta\sigma\,.$$

The first step equation is of the form:

$$(\mu^2+\partial^2)\frac{\delta G^{(1)}}{\delta j(x)}+N\beta\int d^Dy\,\mathrm{Tr}\left[\Big(1-\beta\triangle\cdot\frac{\delta}{i\delta j}\Big)^{-1}(x,y)\triangle(y-x)\right]G^{(1)}= \quad (3.7)$$
$$ij(x)G^{(0)}\,,$$

and we substitute into this equation the following expression for the $G^{(1)}$:

$$G^{(1)}=\Big(\frac{1}{2}j*\mathcal{D}*j+i\tau*j\Big)G^{(0)}\,,$$

where the function $\mathcal{D}(x)$ and the constant $\tau$ are to be determined from equation (3.7). Omitting some straightforward calculations we write down the resulting equation:

$$ij(x) = (\mu^2+\partial^2)\Big((\mathcal{D}*j)(x)+i\tau\Big) - iN\beta^2\int d^Dy(\mathcal{D}*j)(y)\mathrm{Tr}\Big(R(x-y)R(y-x)\Big)$$
$$+N\beta\,\tau\frac{\partial}{\partial(\beta\sigma)}\mathrm{Tr}R(0) - N\beta^3\int d^Dy_1d^Dy_2\mathcal{D}(y_1-y_2)\mathrm{Tr}\Big(R(x-y_1)R(y_1-y_2)R(y_2-x)\Big).$$

From the above relation we immediately find the equation for $\tau$, which defines the first correction to $\sigma$, and the equation for $\mathcal{D}(x)$, which is of the main interest for us:

$$(\mu^2+\partial^2)\mathcal{D}(x) - iN\beta^2\int d^Dy\mathcal{D}(x-y)\mathrm{Tr}\Big(R(-y)R(y)\Big) = i\delta(x)\,. \tag{3.8}$$

Equations (3.6) and (3.8) are still formal due to the ultraviolet divergencies in $\mathrm{Tr}R(0)$ and $\mathrm{Tr}(R\cdot R)$ and we should renormalize our model in order to give them a definite meaning.

Let us consider now the two-dimensional case that is put $D=2$. Analysis of the divergencies of theory (3.2) shows that it is sufficient to introduce only one counter term $\delta\mu^2_{(0)}$ to cancel the divergencies in the equations for $\sigma$ and $\mathcal{D}$. Then the leading approximation equation (3.6) is modified as follows:

$$\sigma = -\frac{N\beta}{\mu^2+\delta\mu^2_{(0)}}\,\frac{M}{2\pi}\ln\left(1+\frac{\Lambda^2}{M^2}\right)\,,$$

where we substitute the value of the $\mathrm{Tr}R(0)$ calculated with a momentum cutoff $\Lambda$.

To verify the possibility of the spontaneous chiral symmetry breaking we should study the above equation at the chiral limit $m\to 0$ ($M\to -\beta\sigma$):

$$\sigma = \frac{N\beta^2}{\mu^2+\delta\mu^2_{(0)}}\,\frac{\sigma}{2\pi}\ln\left(1+\frac{\Lambda^2}{\beta^2\sigma^2}\right)\,. \tag{3.9}$$

The obvious solution to this equation $\sigma=0$ leads (at least up to the few first steps of the scheme) to the usual perturbative expansion of the generating functional with massless fermions. Such a solution is unsatisfactory one from the physical point of view since it contains the tachion states which means that the perturbation is carried over the unstable vacuum [4]. That is why we will concentrate on a possible non zero solution, for which we have:

$$\mu^2+\delta\mu^2_{(0)} = \frac{N\beta^2}{2\pi}\ln\left(1+\frac{\Lambda^2}{\beta^2\sigma^2}\right)\,. \tag{3.10}$$

Now from equation (3.8) we get the following expression for the Fourier image $\mathcal{D}(p^2)$:

$$\mathcal{D}(p^2) = \frac{i}{\mu^2+\delta\mu^2_{(0)}+N\beta^2\Sigma(p^2)-p^2}\,,$$

where the bare mass operator $\Sigma$ reads:

$$\Sigma(p^2)\equiv -i\int\frac{d^Dk}{(2\pi)^D}\,\mathrm{Tr}R(p+k)R(k)\,. \tag{3.11}$$

Let us renormalize $\mathcal{D}(p^2)$ in the Euclidean momentum region by the condition:

$$\text{at}\;\; p^2=-\omega^2 \qquad \mathcal{D}(-\omega^2) = \frac{i}{\mu^2}\,. \tag{3.12}$$

With the renormalization prescription (3.12) we can find the sum $(\mu^2 + \delta\mu^2_{(0)})$ and write the renormalized function $\mathcal{D}$ and the equation for the nonzero value of $\sigma$ at the chiral limit as follows:

$$i\mathcal{D}^{-1}(p^2) = \mu^2 - (p^2 + \omega^2) + \frac{N\beta^2}{2\pi}\left(f\Big(-\frac{p^2}{\beta^2\sigma^2}\Big) - f\Big(\frac{\omega^2}{\beta^2\sigma^2}\Big)\right) \tag{3.13}$$

$$\frac{N\beta^2}{\pi} + \frac{N\beta^2}{2\pi} f\Big(\frac{\omega^2}{\beta^2\sigma^2}\Big) = \mu^2 - \omega^2\,, \tag{3.14}$$

where the function $f(\theta)$ reads:

$$f(\theta) \equiv \int_0^1 \ln(1 + \theta x(1-x))dx\,.$$

In the region $p^2 > 0$, $\mathcal{D}(p^2)$ defines the $s$-channel amplitude of the two fermion scattering, therefore, as directly follows from (3.13), the point $p^2 = 4\beta^2\sigma^2$ is a two particle threshold and a fermion with non zero mass $m_F = \beta\sigma$ exists in our theory.

Now we take the Gross-Neveu limit: $\beta = \mu\sqrt{\lambda}$, $\mu \to \infty$. It can be shown that at this limit the constant $\sigma$ is proportional to the fermion condensate :

$$\sigma \longrightarrow \sqrt{\lambda}\,\frac{<\bar{\psi}\psi>}{\mu} \quad \Rightarrow \quad \beta\sigma \longrightarrow \lambda <\bar{\psi}\psi>\,, \tag{3.15}$$

therefore

$$m_F = \lambda <\bar{\psi}\psi>\ . \tag{3.16}$$

On the other hand the product $(i\beta)^2\mathcal{D}(p^2)$ in the Euclidean region $p^2 = -q^2 < 0$ tends to the running coupling constant of the Gross-Neveu model:

$$(i\beta)^2\mathcal{D}(-q^2) \longrightarrow -i\lambda_r\Big(\frac{q^2}{\omega^2};\frac{m_F^2}{\omega^2}\Big)\,.$$

Calculating the limit explicitly we find the running coupling for the Gross-Neveu model:

$$\lambda_r\Big(\frac{q^2}{\omega^2};\frac{m_F^2}{\omega^2}\Big) = \frac{\lambda}{1 + \frac{N\lambda}{2\pi}\Big(f(q^2/m_F^2) - f(\omega^2/m_F^2)\Big)}\,. \tag{3.17}$$

It is obvious from (3.17) that $\lambda = \lambda_r(1;\frac{m_F^2}{\omega^2})$. Then, denoting the massless combination $m_F^2/\omega^2$ as $\xi$ we find from (3.14):

$$\lambda_r^{-1}(1;\xi) = \frac{N}{\pi}\sqrt{1+4\xi}\,\mathrm{arcsinh}\frac{1}{\sqrt{4\xi}}\,, \tag{3.18}$$

which gives the final expression for the coupling constant (3.17):

$$\lambda_r\Big(\frac{q^2}{m_F^2}\Big) = \frac{\pi}{N\sqrt{1+\frac{4m_F^2}{q^2}}\,\mathrm{arcsinh}\sqrt{\frac{q^2}{4m_F^2}}}\,. \tag{3.19}$$

No free parameters are available except for the fermion mass $m_F$. It is in the full agreement with the number of parameters of the initial Gross-Neveu model: one dimensionless parameter $\lambda$ is changed for one mass parameter $m_F$ [4].

In the deep Euclidean region $q^2 \to \infty$, the running coupling constant asymptotically vanishes

$$\lambda_r\Big(\frac{q^2}{m_F^2}\Big) \sim \frac{\pi}{N \ln \sqrt{\frac{q^2}{m_F^2}}}\,, \tag{3.20}$$

which reflects the known property of the asymptotical freedom of the Gross-Neveu model in the two dimensions.

# 4 D=3

The three dimensional case is interesting in the two aspects. Firstly, the four fermion interaction model (3.1) is not perturbatively renormalizable, since the mass dimension of the coupling $\lambda$ is negative. But the effective Yukawa theory (3.2) is still renormalizable (even super renormalizable) model. Therefore, our approach is a method for handling the non-renormalizable theory. And secondly, the dimensional regularization plays a distinguished role in the three dimensions, due to a typical singularity of a one-loop term of the theory (3.2) has the form $\Gamma(1-D/2)$ or $\Gamma(2-D/2)$ and, therefore, the divergencies are absent in this regularization if $D=3$. The theory is finite and no renormalization is needed. That is why we make the calculations in two regularizations: the dimensional and momentum cutoff ones and compare results.

The form of the main equations (3.6) and (3.8) does not depend on the concrete value of the space-time dimension, so we start our consideration directly with those equations.

*Dimensional regularization*

On calculating the quantity $\mathrm{Tr}R(0)$ in the $D = 3-2\varepsilon$ dimensions we obtain from (3.6) the equation

$$\mu^2\sigma + 3\frac{\Gamma(-1/2)}{(4\pi)^{3/2}} N\beta M^2 = 0\,,$$

which transforms at the limit $m \to 0$ to the following result:

$$\sigma\left(\mu^2 - \frac{3N\beta^3}{4\pi}\sigma\right) = 0\,. \tag{4.1}$$

We have two solutions again: the trivial $\sigma=0$ and the nontrivial $\sigma = \frac{4\pi\mu^2}{3N\beta^3}$ ones. The nontrivial solution gives the nonzero condensate at the Gross-Neveu limit:

$$\beta\sigma \;\longrightarrow\; \lambda <\bar\psi\psi>= \frac{4\pi}{3N\lambda}\,. \tag{4.2}$$

From equation (3.8) we find

$$i\mathcal{D}^{-1}(p^2) = \mu^2 - p^2 + \frac{3N\beta^2}{2\pi}\int_0^1 dx\sqrt{\beta^2\sigma^2 - p^2x(1-x)}\,, \tag{4.3}$$

from which in turn we get the mass of a fermion:

$$m_F = \beta\sigma = \lambda <\bar\psi\psi>= \frac{4\pi}{3N\lambda}\,. \tag{4.4}$$

In this case the running coupling in the Euclidean region $p^2 = -q^2 < 0$ reads:

$$\lambda_r^{-1}(q^2) = \frac{3Nm_F}{4\pi}\left(1 + \frac{m_F}{\sqrt{q^2}}\left(1 + \frac{q^2}{4m_F^2}\right)\arcsin\sqrt{\frac{q^2}{4m_F^2 + q^2}}\right) . \tag{4.5}$$

Its asymptotics in the deep Euclidean region $q^2 \to \infty$ is

$$\lambda_r(q^2) \sim \frac{32}{3N\sqrt{q^2}} ,$$

i. e., the Gross-Neveu model is asymptotically free in the three dimensions.

*The momentum cutoff regularization*

The one-fermion loop $\mathrm{Tr}(R \cdot R)$ and the fermion tadpole $\mathrm{Tr}R(0)$ possess the linear divergence, when calculated at the momentum cutoff $\Lambda$, therefore we introduce a counter term $\delta\mu_{(0)}^2$ to cancel the divergence at the leading approximation. For the nonzero value of $\sigma$ we have the following equation

$$\mu^2 + \delta\mu_{(0)}^2 = \frac{3N\beta^2}{2\pi^2}\left(\Lambda + \frac{\pi}{2}\beta\sigma\right) . \tag{4.6}$$

Equation (3.8) gives:

$$i\mathcal{D}^{-1}(p^2) = \mu^2 + \delta\mu_{(0)}^2 - p^2 + \frac{3N\beta^2}{2\pi}\left(\int_0^1 dx\sqrt{\beta^2\sigma^2 - p^2x(1-x)} - \frac{\Lambda}{\pi}\right) .$$

For the renormalization of the theory we take the same scheme (3.12) as we did in the previous section, i. e., we put

$$\mathcal{D}(-\omega^2) = \frac{i}{\mu^2}$$

at some Euclidean momentum $p^2 = -\omega^2$. We note, that due to the leading approximation equation (4.6) the renormalized function $\mathcal{D}$ does not depend on the concrete regularization scheme and as a consequence so does the running coupling.

Using the renormalization prescription we can define the sum $\mu^2 + \delta\mu_{(0)}^2$ and get the renormalized equation for the nonzero constant $\sigma$ from (4.6). At the Gross-Neveu limit the equation can be written in the form :

$$\sin^2 z + b(z - \sin 2z) = 0 \quad , \quad b \equiv \frac{3N}{8\pi}\lambda\omega , \tag{4.7}$$

where we have denoted

$$z = \arcsin\frac{\omega}{\sqrt{\omega^2 + 4\lambda^2 <\bar{\psi}\psi>}} \qquad z \in [0, \pi/2] ,$$

and have substituted $\beta\sigma \to \lambda <\bar{\psi}\psi>$. One can show that equation (4.7) always has a nonzero solution $z \in [0, \pi/2]$. Under the condition $\omega \ll \lambda <\bar{\psi}\psi>$ the solution is of the form

$$\lambda <\bar{\psi}\psi> = \frac{4\pi}{3N\lambda} .$$

The analytic structure of the renormalized function $\mathcal{D}$ allows us to conclude that there exists a fermion with the mass $m_F = \beta\sigma$ in our model and to get a running coupling constant in the momentum cutoff regularization:

$$\lambda_r^{-1}(q^2) = \frac{3Nm_F}{4\pi}\left(1 + \frac{m_F}{\sqrt{q^2}}\Big(1+\frac{q^2}{4m_F^2}\Big)\arcsin\sqrt{\frac{q^2}{4m_F^2+q^2}}\,\right)\,,$$

which is identical to the result of the dimensional regularization (4.5).

The only parameter of the model is the fermion mass, which defines the strength of interaction at the classical limit $q \to 0$:

$$\lambda_r(0) = \frac{4\pi}{9Nm_F}\,. \tag{4.8}$$

The similar results were obtained in the framework of $1/N$ expansion [7].

# 5 D=4

The case of the four dimensional space-time is more involved due to the effective theory contains the divergencies in the four point Green function of the scalar field. Therefore we have to introduce into the action the corresponding self-interaction of the scalar field $\sim \phi^4$. Thus we start from the action

$$S_{\text{eff}} = \int d^4x \left(\bar\psi\,(i\,\partial\!\!\!/ - m)\,\psi - \frac{1}{2}\phi(Z_\phi\partial^2 + Z_\mu\mu^2)\phi + \beta\,\phi(\bar\psi\psi) - \frac{Z_4\beta^4}{4!}\phi^4\right)\,. \tag{5.1}$$

Here the constants $Z_a$ represent the boson field renormalization whereas the fermion renormalization multipliers are considered as being absorbed into the norm of the fermion fields and into the constants of the action.

The question of equivalence of the model (5.1) (also referred to as "extended Gross-Neveu model") to the Gross Neveu model (3.1) becomes nontrivial in the case $D = 4$. It is evident that such an equivalence can take place in the nonperturbative sense only. This equivalence (in chiral limit) was motivated in the framework of $1/N$-expansion (see, for example [5]) and is based on the fact that in this limit the terms $\phi\,\partial^2\phi$ and $\phi^4$ of effective action (5.1) are irrelevant when one defines physical quantities in the critical infrared region. As will be seen below the same arguing can be applied in our approach too. Therefore we *define* the Gross-Neveu model in $D = 4$ as the Gross-Neveu limit of the renormalized model (5.1)

We will use the dimensional regularization in this section. As the leading approximation to the Schwinger-Dyson equation of the model (5.1) we have the following:

$$\begin{aligned}(Z_\phi^{(0)}\partial^2 + Z_\mu^{(0)}\mu^2)\,\frac{\delta G^{(0)}}{\delta j(x)} - \frac{Z_4^{(0)}\beta^4}{3!}\frac{\delta^3 G^{(0)}}{\delta j^3(x)} &+ \\ N\beta\int d^4y\,\mathrm{Tr}\left[\Big(1-\beta\,\triangle\cdot\frac{\delta}{i\delta j}\Big)^{-1}(x,y)\,\triangle\,(y-x)\right]G^{(0)} &= 0\end{aligned} \tag{5.2}$$

The solution to this equation is sought for in the form $G^{(0)} = \exp(i\sigma * j)$, which gives the connection of the leading counter terms:

$$Z_\mu^{(0)}\mu^2\sigma + \frac{Z_4^{(0)}}{3!}\beta^4\sigma^3 = iN\beta\mathrm{Tr}R(0)\,. \tag{5.3}$$

The principal difference with the previous cases $D = 2, 3$ is that we have *two* counter terms in (5.3) and upon their fixing equation (5.3) turns into an identity and does not define any specific value of $\sigma$. We will seek for the first step approximant in the same form as in the previous sections and find the following result for the Fourier image $\mathcal{D}(p^2)$:

$$i\mathcal{D}^{-1}(p^2) = Z_\mu^{(0)}\mu^2 + Z_4^{(0)}\frac{\beta^4\sigma^2}{2} + N\beta^2\Sigma(p^2) - Z_\phi^{(0)}p^2\,, \tag{5.4}$$

where the mass operator $\Sigma$ is one-fermion loop (3.11). Choose the renormalization prescription for $\mathcal{D}$ at an Euclidean point $p^2 = -\omega^2$ :

$$\text{at}\quad p^2 \simeq -\omega^2 \qquad i\mathcal{D}^{-1}(p^2) = \mu^2 + O((p^2+\omega^2)^2)\,.$$

This prescription fixes the value of the counter terms $Z_\phi^{(0)}$ and the sum $Z_\mu^{(0)}\mu^2 + Z_4^{(0)}\beta^4\sigma^2/2$. Taking into account equation (5.3) allows one to define all the leading approximation counter terms.

Let us consider the chiral limit $m \longrightarrow 0$. First of all it is worth mentioning that equations (5.3) and (5.4) up to notations coincide with analogous equations of $1/N$-expansion method. Therefore the above mentioned arguments of [5] about the equivalence are valid for our consideration too.

We note now that there exist two different cases which do not contradict our equations: the first of them is $\sigma = 0$ and the second one is $\sigma \neq 0$. Let us begin with the case of the nonzero $\sigma$ and take for simplicity $\omega = 0$. The renormalized function $\mathcal{D}$ has the form

$$i\mathcal{D}^{-1}(p^2) = \mu^2 - \frac{N\beta^2}{8\pi^2}\left(p^2 + 6m_F^2\Phi\Big(-\frac{p^2}{m_F^2}\Big)\right), \tag{5.5}$$

where the function $\Phi(\theta)$ stands for the integral:

$$\Phi(\theta) = \int_0^1 (1+\theta x(1-x))\ln(1+\theta x(1-x))dx$$

and $m_F^2 = \beta^2\sigma^2$. On putting as in the above sections

$$(i\beta)^2\mathcal{D}(-q^2) \;\longrightarrow\; -i\beta_r(q^2)$$

we obtain from (5.5) the expression for the Yukawa running coupling $\beta_r(q^2)$

$$\beta_r(q^2) = \frac{\beta_r(0)}{1 + \frac{N\beta_r(0)}{8\pi^2}\left(q^2 - 6m_F^2\Phi(\frac{q^2}{m_F^2})\right)}\,. \tag{5.6}$$

Since equations (5.3) are fulfilled identically, the parameter $\beta_r(0)$ is as free as the mass $m_F$ is.

Now let us investigate the behaviour of the denominator of expression (5.6) in the regions of small and large values of $q^2$. It can be easily seen that at $q^2 \to 0$ this denominator tends to the unity while in the deep Euclidean region $q^2 \to \infty$ its asymptotics is $-q^2\ln q^2 \to -\infty$. Due to the denominator being a continuous function there exists *a finite* value $q_0^2$ at which the running coupling constant has a pole. This indicates to the presence of a tachion state in our model. The only way to save the situation is to put the free parameter $\beta_r(0)$ be equal to zero.

The same is true for another phase of the model, when $\sigma = 0$. The only difference is that we must keep an arbitrary nonzero value of the renormalization point $\omega$ in order to avoid the infrared singularities at the chiral limit. The running coupling reads:

$$\beta_r(q^2) = \frac{\beta_r(\omega^2)}{1 + \frac{N\omega^2\beta_r(\omega^2)}{8\pi^2}F(q^2/\omega^2)}\,, \tag{5.7}$$

where $F(x) = x(1 - \ln x) - 1$. The function $F(x)$ takes its maximum at the point $x = 1$: $F_{max} = 0$ and is negative for all other positive $x$. Therefore we come to the same conclusion as above:

**(i)** if $\frac{8\pi^2}{N} \le \omega^2\beta_r(\omega^2)$, two tachion states are present and the model is contradictory.

**(ii)** if $0 < \omega^2\beta_r(\omega^2) < \frac{8\pi^2}{N}$, one tachion state is present and the model is contradictory again.

**(iii)** if $\beta_r(\omega^2) = 0$, the tachion is absent but the running coupling is identical zero.

From the analysis above it follows that the renormalized effective coupling constant of Yukawa interaction $\phi\bar{\psi}\psi$ should be zero. This means that the renormalized model (5.1) describes a free fermion and a scalar boson with selfinteraction $\phi^4$ decoupled from each other due to trivialization of Yukawa coupling. Further, the triviality of the theory $\phi^4$ in $D = 4$ is well known and was demonstrated in various nonperturbative approaches (see for example [3] and references therein). It is worth pointing out that this triviality can be shown by means of the same method we are using in the present work [3]. In the Gross-Neveu limit $\mu \to \infty$ the free boson disappears from the spectrum and we come to the theory of free fermion.

Note, that the triviality of four-fermion interaction in $D = 4$ was also found in the framework of $1/N$-expansion and confirmed by lattice simulations [8]. Our calculation is one more argument in favor of such a trivialization.

# 6 The bilocal source

In this section we would like to discuss another approach to the Gross-Neveu model. The approach is based on the bilocal source formalism, which is more natural and informative than that of the preceding sections. We will directly deal with the fermion degrees of freedom without integrating over them. This allows one to get the full information on the fermion dynamics. Namely it is possible to calculate the fermion Green functions like the propagator, the amplitude and so on. Besides, this method is technically simpler. But as a price for the above advantages the use of the bilocal source encounters a problem, which makes the results obtained in this formalism to be not very firm from the position of the mathematical rigour. The problem originates from the well known fact that keeping the right Bose or Fermi statistical properties of the Green functions is a rather nontrivial task, when the bilocal source is used. We postpone the discussion of this difficulty to the end of the section.

Now let us demonstrate how the results of the previous sections can be obtained in the bilocal source formalism. We consider the two dimensional $N$-component Gross-Neveu model with the action $S_{GN}$ (3.1) and introduce a bilocal source — the function $\eta_{\alpha\beta}(x, y)$ depending on two space-time points and two multi indices $\alpha$ and $\beta$ including the color and Lorenz degrees of freedom. The $n$-th derivative of the generating functional

$$G(\eta) = \int \mathcal{D}\bar{\psi}\mathcal{D}\psi \exp(iS_{GN} - i\bar{\psi} * \eta * \psi)$$

defines the $2n$-th point Green function. This is an advantage of the bilocal source, since to find a Green function one has to calculate half as many derivatives than when a local source is used. The Schwinger-Dyson equation for the functional $G(\eta)$ is of the form:

$$i\lambda \frac{\delta^2 G}{\delta\eta_{\beta\alpha}(yx)\delta\eta_{\gamma\gamma}(xx)} + (i\,\partial\!\!\!/_x - m)_{\alpha\gamma}\frac{\delta G}{\delta\eta_{\beta\gamma}(yx)} + \delta_{\alpha\beta}\delta(x-y)G \qquad (6.1)$$
$$= \int d^2 z \eta_{\alpha\gamma}(xz)\frac{\delta G}{\delta\eta_{\beta\gamma}(yz)}\,,$$

where the summation over the repeated indices is assumed. The leading approximation $G^{(0)}$ obeys (6.1) with zero right site. Substituting $G^{(0)} = \exp\{\mathrm{Tr}(\eta * \triangle)\}$, we find the equation for the leading approximation to the fermion propagator $\triangle$:

$$(M - i\,\partial\!\!\!/)\triangle(x) = \delta(x)\,, \qquad (6.2)$$

where the mass parameter $M$ denotes the combination:

$$M = m - i\lambda\,\mathrm{Tr}\triangle(0)\,. \qquad (6.3)$$

Since $\mathrm{Tr}\triangle(0)$ is a function of M, relation (6.3) is a consistency condition, which is an analog of the gap equation.

To solve the next step equation

$$i\lambda\frac{\delta^2 G^{(1)}}{\delta\eta\delta\eta} + (i\,\partial\!\!\!/ - m)\frac{\delta G^{(1)}}{\delta\eta} + G^{(1)} = \eta * \frac{\delta G^{(0)}}{\delta\eta}\,, \qquad (6.4)$$

we substitute the expression

$$G^{(1)} = \Big(\frac{1}{2}\mathrm{Tr}_{(12)}F_{12} * \eta_1 * \eta_2 + \mathrm{Tr}\triangle^{(1)} * \eta\Big)G^{(0)}\,, \qquad (6.5)$$

where the function $F^{\alpha_1\alpha_2}_{\beta_1\beta_2}(x_1x_2|y_1y_2)$ is connected with the leading approximant for the two-fermion amplitude and $\triangle^{(1)}$ is the first correction to the propagator. The equations for $F$ and $\triangle^{(1)}$ can be easily obtained from (6.4) and (6.5), so we directly write down the answer for the amplitude omitting the explicit form of the mentioned equations:

$$F^{\alpha_1\alpha_2}_{\beta_1\beta_2}(x_1x_2|y_1y_2) =$$
$$\int dz_1dz_2\Big(\triangle(x_1-z_1)\cdot\triangle(z_1-y_1)\Big)_{\alpha_1\beta_1}\mathcal{K}(z_1-z_2)\Big(\triangle(x_2-z_2)\cdot\triangle(z_2-y_2)\Big)_{\alpha_2\beta_2} \qquad (6.6)$$
$$+\ \dots\,,$$

where the dots stand for the disconnected part.

The scalar kernel $\mathcal{K}$ is a solution of the equation

$$\int dy(1 - i\lambda\Sigma)(x-y)\mathcal{K}(y) = -i\lambda\delta(x) \qquad (6.7)$$

and the function $\Sigma(x)$ is a fermion loop (compare with Section 3)

$$\Sigma(x) = \mathrm{Tr}\triangle(x)\triangle(-x)\,.$$

Equation (6.7) can be easily solved in the momentum space and the Fourier image of the bare kernel $\mathcal{K}$ turns out to be:

$$\mathcal{K}(p^2) = -\frac{i\lambda}{1 - i\lambda\Sigma(p^2)}\,. \qquad (6.8)$$

Let us consider the chiral limit $m \to 0$ of our model. To renormalize the amplitude at this stage we should introduce only one counter term $\delta\lambda_{(0)}$. We require the connected part of the amplitude at the symmetric point in the Mandelstam variables to be:

$$F^{\mathrm{conn}}(s = t = -\omega^2) \equiv \mathcal{K}(-\omega^2) = -i\lambda_r \,, \tag{6.9}$$

where $s$ and $t$ are the Mandelstam variables. This allows us to fix the counter term $\delta\lambda_{(0)}$ and to obtain the renormalized connected amplitude

$$\mathcal{K}^{\mathrm{ren}}(p^2) = -\frac{i\lambda_r}{1 + i\lambda_r(\Sigma(-\omega^2) - \Sigma(p^2))} \,. \tag{6.10}$$

Now we should go back to the consistency condition (6.3) which takes the following form at the chiral limit $m \to 0$

$$M = -i(\lambda + \delta\lambda_{(0)})\,\mathrm{Tr}\,\triangle\,(0) \,.$$

Taking into account the value of $\delta\lambda_{(0)}$ fixed by the renormalization scheme (6.9) we find an equation defining possible values of the fermion mass in our model. Besides the trivial solution $M = 0$, there exists the nontrivial one $M = m_F \neq 0$ which corresponds to the spontaneous symmetry breaking and gives a connection among the fermion physical mass $m_F$, the renormalized coupling $\lambda_r$ and the subtraction point $\omega^2$ (compare with (3.18):

$$N\sqrt{1 + \frac{4m_F^2}{\omega^2}}\,\mathrm{arcsinh}\sqrt{\frac{\omega^2}{4m_F^2}} = \frac{\pi}{\lambda_r} \,. \tag{6.11}$$

With the help of (6.11) we can get the running coupling constant from (6.10)

$$\lambda_r\Big(\frac{q^2}{m_F^2}\Big) = \frac{\pi}{N\sqrt{1 + \frac{4m_F^2}{q^2}}\,\mathrm{arcsinh}\sqrt{\frac{q^2}{4m_F^2}}} \,, \tag{6.12}$$

which is identical to (3.19).

Turning to the discussion at the beginning of this section, where is a vulnerable place of the above consideration? It is contained in the first term of the Schwinger-Dyson equation (6.1). Indeed it can be easily shown that due to the Fermi statistics of the spinor fields the generating functional $G(\eta)$ obeys identically the relation:

$$\frac{\delta^2 G}{\delta\eta_{\beta_1\alpha_1}(y_1x_1)\delta\eta_{\beta_2\alpha_2}(y_2x_2)} = -\frac{\delta^2 G}{\delta\eta_{\beta_1\alpha_2}(y_1x_2)\delta\eta_{\beta_2\alpha_1}(y_2x_1)} \,. \tag{6.13}$$

Therefore we can write the first term of (6.1) as

$$-\frac{\delta^2 G}{\delta\eta_{\beta\gamma}(yx)\delta\eta_{\gamma\alpha}(xx)} \tag{6.14}$$

and the equation thus obtained will be equivalent to (6.1) from the point of view of the *full* generating functional which is a strict solution of the Schwinger-Dyson equation. But for our approximant $G^{(0)}$ such a change has a crucial consequence since it does not possess the property (6.13). And the substitution (6.14) leads to the appearance of the term $\triangle_{\alpha\gamma}(x)\,\triangle_{\gamma\beta}\,(0)$ instead of $\triangle_{\alpha\beta}(x)\mathrm{Tr}\,\triangle\,(0)$ in equation (6.2). But such an equation for $\triangle(x)$ does not possess any solution except for the perturbative one. The possible way to resolve this contradiction is in the use of properly symmetrized form of the first term in equation (6.1). Such a new

equation will be still equivalent to (6.1) for the full generating functional and will respect the property (6.13), when the approximate solutions is used.

It can also be shown that the important property (6.13) which is responsible, in particular, for the crossing symmetry is restored for the Green functions successively step by step of the approximation scheme. For example, all the Green functions calculated from the $G^{(0)}$ will consist of disconnected parts only and will not possess the true structure demanded by the Fermi statistics. But taking into account the first correction $G^{(1)}$ not only gives the leading approximation to the connected part of the amplitude but also restores the true statistical structure for the disconnected part of the four point Green function approximant. At the next steps we will successively find corrections improving the connected and disconnected parts of the higher Green functions. The use of the symmetrized form of the Schwinger-Dyson equation allows one to get the properly symmetrized Green functions directly at the corresponding steps of the scheme.

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## References

[1] P.M. Stevenson, *Phys. Rev.* **D30**, 1712 (1984).
P.M. Stevenson, *Phys. Rev.* **D32**, 1389 (1985).

[2] A. Sissakian, I. Solovtsov and O. Shevchenko, *Int. J. Mod. Phys.* **A9**, 1929 (1994).

[3] V.E. Rochev, *J. of Phys. A: Math. Gen.* **30**, 3671 (1997), hep-th/9606155.

[4] D.J. Gross and A. Neveu, *Phys. Rev.* **D10**, 3235 (1974).

[5] J. Zinn-Justin, *Quantum Field Theory and Critical Phenomena*, (Clarendon Press, Oxford, second ed., 1993).

[6] K.G. Klimenko, *Z.Phys.* **C37**, 457 (1988).

[7] B. Rosenstein, B.J. Warr and S.H. Park, *Phys. Rep.* **205**, 59 (1991).

[8] S. Kim, A. Kocić and J. Kogut, *Nucl. Phys.* **B429**, 407 (1994).

[9] V.E. Rochev and P.A. Saponov, in *Quanta, Relativity, Gravitation*, Proc. of XVIII Workshop on Quantum Field Theory and High Energy Physics, June 26–30, 1995, Protvino, Russia, eds. V.A. Petrov *et al.*, (IHEP, Protvino, 1996) p. 283.